**D** An ODE is called autonomous if the independent variable does not appear explicitly.

\[
\frac{dy}{dt} = f(y)
\]

This is separable, however sometimes we may not be able to integrate, for example:

\[
\frac{dy}{dt} = e^{yt}
\]

When this happens, we resort to the geometric viewpoint. It is often useful even if we can integrate.

**Exponential Growth**

\[
\begin{align*}
\frac{dy}{dt} &= ry \\
y(0) &= y_o
\end{align*}
\]

\[y(t) = y_0 e^{rt}, \text{ but this is unrealistic as } t \to \infty.\] (Food constraints, for example)

**Logistic Growth**

\[
\frac{dy}{dt} = h(y) y,
\]

choose \( h(y) \) st. \( h(y) \approx r \) when \( y \) is small, but decreases as \( y \) grows, and \( h(y) \to 0 \) for \( y \) sufficiently large.

ex: \( h(y) = r - ay \),

\[
\frac{dy}{dt} = (r-ay) y = r \left(1 - \frac{y}{K}\right) y \quad \text{(logistic equation)}
\]

INTRINSIC GROWTH RATE

**D** An equilibrium solution \( y = g(t) \) is a solution with \( \frac{dy}{dt} = g'(t) = 0 \).

\[r \left(1 - \frac{y}{K}\right) y = 0 \Rightarrow y = 0 \text{ or } y = K.
\]

We attempt to plot solution curves, using information about the derivative.

We can get concavity information from

\[
\ddot{y} = \frac{d^2y}{dt^2} = \frac{d}{dy} \left( \frac{dy}{dt} \right).
\]
We can get concavity information from
\[ \frac{d^2 y}{dt^2} = \frac{d}{dt} \left( \frac{dy}{dt} \right) = \frac{d}{dt} \left( f(y) \right) \]

\[ = f'(y) \frac{dy}{dt} = f'(y)f(y). \]

\[ \frac{dy}{dt} = r \left( 1 - \frac{y}{K} \right) y = f(y) \Rightarrow f'(y) = -\frac{r}{K} y + r \left( 1 - \frac{y}{K} \right) = r \left( 1 - \frac{2y}{K} \right) \]

\[ \begin{array}{c}
\text{Solutions} \\
\text{start in} \quad (0, K) \quad \text{we call it the saturation level} \end{array} \]

\[ K \text{ is the upper bound which is approached, but not exceeded, by all} \]

\[ \text{solutions starting in} \quad (0, K) \quad \text{we call it the saturation level} \]

of a species.

Let's solve it:
\[ \frac{dy}{dt} = r \left( 1 - \frac{y}{K} \right) y \]
\[ \Rightarrow \int \frac{dy}{(1 - \frac{y}{K})y} = \int r dt = rt + C \]

Recall partial fractions:
\[ \frac{1}{r \left( 1 - \frac{y}{K} \right) y} = \frac{A}{y} + \frac{B}{1 - \frac{y}{K}} \]

Integrating,
\[ \Rightarrow \ln |y| - \ln \left| 1 - \frac{y}{K} \right| = rt + C \]
\[ \Rightarrow \frac{y}{1 - \frac{y}{K}} = Ce^{rt} \]

Let \( y(0) = y_0 \), \( C = y_0 / (1 - \frac{y_0}{K}) \).

Solving for \( y \):
\[ y = Ce^{rt} \left( 1 - \frac{y_0}{K} \right) \]
\[ \Rightarrow y(1 + \frac{C}{K}e^{rt}) = Ce^{rt} \]
\[ \Rightarrow y = \frac{Ce^{rt}}{1 + \frac{C}{K}e^{rt}} = \frac{\left( \frac{y_0}{K} \right) e^{rt}}{1 + \frac{C}{K}e^{rt}} = \frac{y_0}{\frac{y_0}{K}e^{rt} + (k-y_0)e^{rt}} \]
Note $y=0$, $y=K$ are contained in $T$

Note also that if a solution starts near $K$,
\[
\lim_{t \to \infty} y(t) = K.
\]

For this reason, $y(t) = K$ is called an **asymptotically stable solution**.

For solutions starting near 0, however, they move away from 0, so we call $y(t) = 0$ an **unstable solution**.

Much of the qualitative information can be found simply from the sign of $\frac{dy}{dt}$.

\[
\frac{dy}{dt} = -r \left(1 - \frac{y}{K}\right) y =: f(y)
\]

**Equilibrium points**: $\frac{dy}{dt} = 0 \Rightarrow y = 0$ or $y = K$.

$f'(y)$ $\uparrow$ concavity $\downarrow$ $\uparrow$ $\downarrow$ $\downarrow$ $\uparrow$ $\downarrow$ $\downarrow$

$\begin{align*}
    f'(y) &= \frac{dy}{dt} = -r \left(1 - \frac{y}{K}\right) = -r \left(1 - \frac{K}{y}\right) \\
    &= -r \left(1 - \frac{y}{K}\right)
\end{align*}$

Since population does not survive unless $y_0 > T$, we call $T$ a **threshold**.

Another way: inspect graph of $f(y)$:
Note: In homework for § 2.5, question 17a, you have to solve

\[
\frac{dy}{dt} = ry \ln \left( \frac{K}{y} \right) = -ry \ln \left( \frac{y}{K} \right)
\]

Hint: Let \( u = \ln \left( \frac{y}{K} \right) \), \( u' = \frac{y'}{K} = \frac{y'}{y} \), ...